A model for physiological transmembrane transport derived from thermodynamical principles

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Supplementary File 1

The Goldman constant field approximation from the general formulation

Let $x = [x]_j$, $j \in \{0,1\}$ (extra, and intracellular concentrations of an ion of type x. The Goldman-Hodgkin-Katz equation describing the transmembrane current carried by x-ions is given by

$$\begin{aligned}
I_x &= g_x\, v \left[\frac{x_0 - x_1 \exp\left(z_x \frac{v}{v_T}\right)}{1-\exp\left(z_x \frac{v}{v_T}\right)}\right] \\
&= g_x\, v \left[\frac{\exp\left((b-1)z_x \frac{v}{v_T}\right)\left(x_1 \exp\left(z_x \frac{v}{v_T}\right) - x_0\right)}{\exp\left((b-1)z_x \frac{v}{v_T}\right)\left(\exp\left(z_x \frac{v}{v_T}\right)-1\right)}\right] \\
&= g_x\, v \left[\frac{x_1 \exp\left(bz_x \frac{v}{v_T}\right) - x_0 \exp\left((b-1)z_x \frac{v}{v_T}\right)}{\exp\left(bz_x \frac{v}{v_T}\right) - \exp\left((b-1)z_x \frac{v}{v_T}\right)}\right] \\
&= g_x\, v \left(\frac{x_1^{b-1}\, x_0^{-b}}{x_1^{b-1}\, x_0^{-b}}\right)\left[\frac{x_1 \exp\left(bz_x \frac{v}{v_T}\right) - x_0 \exp\left((b-1)z_x \frac{v}{v_T}\right)}{\exp\left(bz_x \frac{v}{v_T}\right) - \exp\left((b-1)z_x \frac{v}{v_T}\right)}\right] \\
&= g_x\, v \left(x_1^{1-b}\, x_0^{b}\right)\left[\frac{\left(\frac{x_1}{x_0}\right)^b \exp\left(bz_x \frac{v}{v_T}\right) - \left(\frac{x_0}{x_1}\right)^{1-b} \exp\left((b-1)z_x \frac{v}{v_T}\right)}{\exp\left(bz_x \frac{v}{v_T}\right) - \exp\left((b-1)z_x \frac{v}{v_T}\right)}\right] \\
&= g_x\, v \left(x_1^{1-b}\, x_0^{b}\right)\left[\frac{\exp\left(bz_x \frac{v-v_x}{v_T}\right) - \exp\left((b-1)z_x \frac{v-v_x}{v_T}\right)}{\exp\left(bz_x \frac{v}{v_T}\right) - \exp\left((b-1)z_x \frac{v}{v_T}\right)}\right] \\
&= A_x \left[\exp\left(bz_x \frac{v-v_x}{v_T}\right) - \exp\left((b-1)z_x \frac{v-v_x}{v_T}\right)\right]
\end{aligned} \tag{A1}$$

where

$$A_x = \left[\frac{g\ v\ x_1^{1-b}\ x_0^b}{\exp\left(bz_x \frac{v}{v_T}\right) - \exp\left((b-1)z_x \frac{v}{v_T}\right)} \right] \tag{A2}$$

is an amplitude term that can be approximated by a constant (Nonner et al., 1998; Nonner and Eisenberg, 1998; Endresen et al., 2000). A specific example for a calcium current at 24^{o}C can be found in equations 25 and 26 in the article by Herrera-Valdez and Lega (2011). Notice that equations (A1)-(A2) reduce to

$$I_x = \left[\frac{g\ v\ (x_1 x_0)^{1/2}}{\exp\left(\frac{z_x v}{2v_T}\right) - \exp\left(z_x \frac{v}{2v_T}\right)} \right] \left[\exp\left(z_x \frac{v - v_x}{2v_T}\right) - \exp\left(z_x \frac{v - v_x}{2v_T}\right) \right] = \left[\frac{g\ v\ (x_1 x_0)^{1/2}}{\sinh\left(\frac{z_x v}{2v_T}\right)} \right] \sinh\left(z_x \frac{v - v_x}{2v_T}\right) \tag{A3}$$

when $b = 1/2$.

Fast spiking interneuron dynamics

A simple model of the dynamics of a fast spiking (FS) striatal interneuron can be constructed using (**??**). To do so, assume the transmembrane potential depends on three currents respectively mediated by Na-K pumps, non-inactivating K^+ channels, and Na^+ channels with transient dynamics, with voltage-dependent gating in both channels. It is also assumed that the proportion of activated K^+ is represented by a variable $w \in [0, 1]$, which also represents the proportion of inactivated Na^+ channels (Av-Ron et al., 1991; Rinzel, 1985). That is, $1-w$ represents the proportion of non-inactivated Na^+ channels. The dynamics for w can be assumed to follow a logistic scheme, capturing the behaviour of delayed-rectifier K^+ currents typically recorded in voltage clamp mode without adding extra powers to w (see for instance Hodgkin and Huxley, 1952, and the Appendix). It is also assumed that sodium channel activation is fast, described by its steady state function of v.

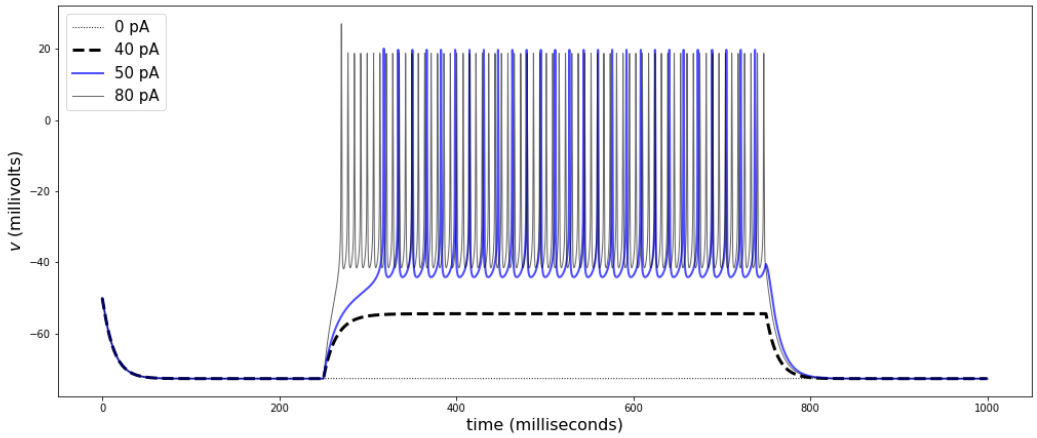

Figure A1. Rest to spiking transitions of FS interneuron under current clamp. The traces show responses to current-clamp stimulation of different amplitudes. The transition between rest and spiking with a rheobase occurs between 40 and 50 pA, as shown for some FS neurons in the mouse striatum (Orduz et al., 2013). The traces correspond to stimulation amplitudes of 0 (gray dots), 40 (black dashed line), 50 (blue), and 80 pA (gray). Parameters can be found in Table A2.

Explicitly, the dynamics of the system can then be captured by coupled differential equations of the form

$$\partial_t v = -(1-w)F_m(v)a_{\mathrm{NaT}}\ \psi_{\mathrm{NaT}}(v) - w a_{\mathrm{KaD}}\ \psi_{\mathrm{KaD}}(v) - a_{\mathrm{NaK}}\ \psi_{\mathrm{NaK}}(v), \tag{A4}$$

$$\partial_t w = w[F_w(v) - w]R_w(v), \tag{A5}$$

The activation rate for K^+ channels depends is a voltage-dependent functions R_w and F_w as defined for the cardiac pacemaking model.
The dynamics of the system are such that, as v increases, w increases, but at a slower rate in comparison to v. This is because the activation w is always moving toward its steady state value, which increases as v increases. Once w increases, the Na^+ current tends to decrease and the K^+ current tends to increase, thereby causing a decrease in v. The slower dynamics in w relative to those in v capture the delay between the amplification caused by the Na^+ current and the recovery caused by the K^+ current. The current mediated by Na/K-ATPase acts as an extra attracting force toward v_{NaK} that increases nonlinearly as the distance between v and v_{NaK} increases.
Striatal FS interneurons display maximum $\partial_t v$ between 100 and 200 V/s. In current clamp mode, most neurons are silent, and show transitions between rest and repetitive spiking at a rheobase current of approximately 90 pA, with initial firing rates between 50 and 60 Hz and a delay to first spike in the transition that decreases as the stimulus amplitude increases (Fig.A1, parameters in Table A2).
To include these properties into the model, the membrane capacitance was specified first, then the maximum $\partial_t v$ was adjusted by fitting the parameter a_{NaT}, and then the contributions for the K^+ channels and the Na-K ATPase are set to obtain spiking and fit the rheobase.
The model in equations (A4)-(A5) reproduces dynamics observable in fast spiking neurons in CA1 (Erisir et al., 1999) or in the striatum (Orduz et al., 2013; Tepper et al., 2010).

Table A1. Physical constants. The conversion factor f from pA to $\mu M \cdot ms^{-1} = mM \cdot s^{-1}$ implies $\Rightarrow M = f \cdot 10^{-9}$ Coul. Then $f = 10^9$ M/Coul $= 10^9 \cdot 96485.3329/F \approx 10^{14}$ M/Coul.

Constant	Value	Units	Description
T_0	273.16	degrees Kelvin	Zero absolute temperature
N_A	6.023×10^{23}	molecules/Mole	Avogadro's number
q	$1.60217733 \times 10^{-19}$	Coulombs/molecule	Elementary charge
k	1.381×10^{-23}	Joules/^{o}Kelvin	Boltzmann constant
$F = qN_A$	96485.33289	Coulombs/Mole	Faraday's constant (the magnitude of electric charge per mole of electrons)
$R = kN_A$	1.987	cal/(Mole ^{o}Kelvin)	Gas constant

Table A2. Parameters for the fast spiking interneuron model.

Parameter	Value	Units	Description
Current amplitudes and capacitance for the neuronal membrane model			
C_m	30	pF	Membrane capacitance
$\bar{a}_{\mathrm{NaK}}$	67	pA	Maximum amplitude for the Na^+-K^+ ATPase current
$\bar{a}_{\mathrm{K}}$	4400	pA	Maximum amplitude for the delayed-rectifier K^+ current
$\bar{a}_{\mathrm{Na}}$	1400	pA	Maximum amplitude for the transient Na^+ current
v_{ATP}	-430	mV	Potential ATP hydrolysis
$v_{\mathrm{NaK}} = 3v_{Na} - 2v_{\mathrm{K}} + v_{\mathrm{ATP}}$	-72	mV	Reversal potential for the for Na^+-K^+ ATPase current
v_{K}	-89	mV	Nernst potential for K^+
v_{Na}	60	mV	Nernst potential for Na^+
v_{mT}	-17	mV	Half-activation potential for the transient Na^+-current
v_w	-5	mV	Half-activation potential for the transient K^+-current
g_{mT}	5	–	Activation slope factor for the transient Na^+-current
g_w	4	–	Activation slope factor for the K^+-current
r_w	2	s^{-1}	Activation rate for the neuronal K^+-current
b_w	0.3	–	Activation slope factor for the K^+-current
b_{NaK}	0.5	–	Non-rectification for the Na^+-K^+-current
b_{K}	0.5	–	Non-rectification for the transient K^+-current
b_{Na}	0.5	–	Non-rectification for the transient Na^+-current

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